TERRABRAINS RESEARCH

The Collapse Load of a Non-Associated Frictional Soil

What can be proved about the bearing capacity of a strip footing when the soil dilates less than it rubs

TerraBrains Research · Technical Report TR-2026-01 · 1 November 2026

ABSTRACT

For a smooth rigid strip on a weightless, cohesionless Mohr–Coulomb soil with friction angle $\varphi' = 25°$ and zero dilatancy ($\psi = 0$) — the benchmark of Potts & Zdravković (2001) — we certify $1 \le N_q \le 10.74$. The upper bound is a finite-element limit-analysis bound for the associated material, 0.68 % above Prandtl's exact 10.662; the lower bound is Radenković's theorem. We prove that no comparison material admitted by that theorem can raise the lower value: for $c = 0$ and $\psi = 0$ every admissible one is a liquid. We prove two conditional lower bounds — $N_q \ge 1 + (2 + \pi) \sin \varphi \cdot p_{\min}/q_0$ if the mean effective stress stays above $p_{\min}$ wherever the soil deforms, and Davis's 8.58 if collapse occurs by statically determinate translating blocks — and state why neither is claimed for this footing. Published finite-element values (≈ 10.0) and TerraBrains' own estimates (10.3–10.6) lie inside the bracket; none is certified. An assumption-free lower bound above 1 remains open.

KEYWORDS bearing capacity · non-associated plasticity · dilatancy · limit analysis · Radenković's theorem · Mohr–Coulomb · strip footing · uniqueness

$1 \le N_q \le 10.74$	10.66	8.58	≈ 10.0 – 10.6
certified bracket, ψ = 0	exact value with associated flow (Prandtl)	Davis – a bound only if collapse is by sliding blocks	finite-element estimates (published and ours)

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Status words are used in one sense only. **Certified**: follows from a theorem, with any computation done by a method that preserves the property the theorem needs. **Conditional**: proved under a stated hypothesis that is not known to hold for the real collapse. **Estimate**: a computed or fitted number with no proof of where it lies relative to the true collapse load. **Reference**: an exact value for a different (associated) material.

1 Introduction

1.1 The question

The bearing-capacity factor N_q familiar to every geotechnical engineer is derived for a material whose plastic strain rate is normal to its yield surface — associated flow, in which the dilatancy angle ψ equals the friction angle φ. For soils, "the dilatancy angle is known to be significantly smaller than the friction angle" (Vermeer & de Borst, 1984, p. 6). Once $\psi < \varphi$, the classical theory loses its strongest result. Salençon (1977, Ch. V §8, p. 119) puts it directly: the uniqueness theorem of the limit loadings is no longer applicable, and "it may be possible for the same loading to be a limit loading for the elasto-plastic system along one loading path, but not along another". Vermeer & de Borst (1984, §5.3, pp. 29–30) show computed limit loads that depend on the initial stresses and on the sequence of loading, and (Ch. 8, pp. 45–51) bifurcation into shear bands while the hardening modulus is still positive.

Radenković's note of 1961, which founded the limit theory for these materials, says in its third paragraph that his bounds do not justify a uniqueness statement, "at least without additional data", while expecting unique limit loads in statically determinate problems. The strip footing is not covered by that exception: with $\psi \neq \varphi$ the Prandtl stress field admits no velocity field obeying the flow rule with a non-negative multiplier (Salençon, 1977, §8.4, p. 121; Drescher & Detournay, 1993, Appendix 3). We therefore say that the collapse load of a non-associated soil *need not be* unique. We do not claim that it is non-unique; we claim only that no theorem guarantees one value. A finite-element program will always print a number. The question of this report is which statements about that number can be proved.

1.2 The benchmark

We use the strip-footing problem of Potts & Zdravković (2001, Vol. 2, §6.5.4.4, pp. 225–227), chosen because it isolates the effect of the flow rule: the soil is weightless and cohesionless, so the only strength is frictional and the only load scale is the surcharge. Table 1 lists the data and Figure 1 the geometry. The bearing-capacity factor is $N_q = q/q_0$, the collapse pressure divided by the surcharge; for a weightless soil it does not depend on the footing width.

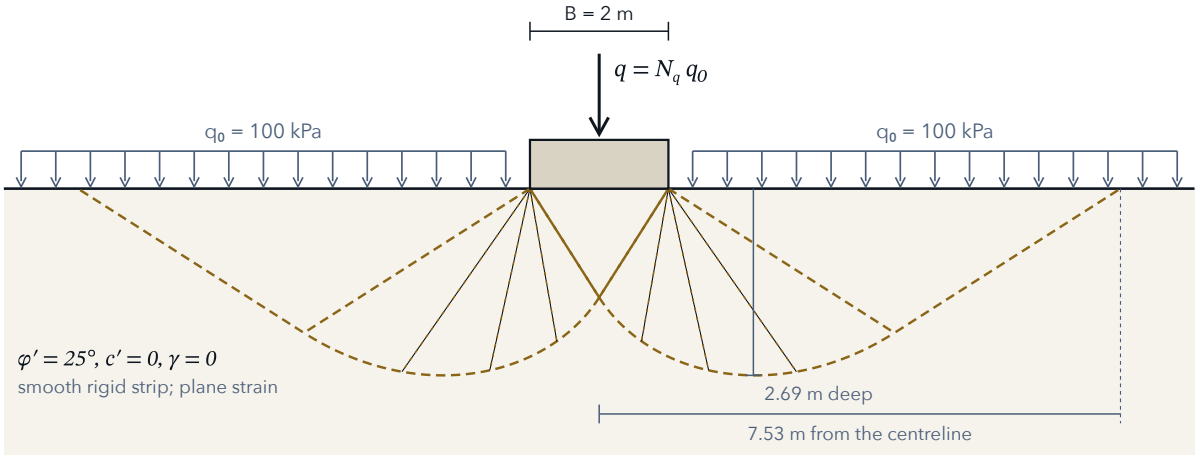


Figure 1. The benchmark problem. The dashed lines are Prandtl's collapse mechanism for the *associated* material ($\psi = \varphi$) – an active wedge under the footing, a logarithmic-spiral fan and a passive wedge – whose load, N_q = 10.662, is exact for that material. For B = 2 m it reaches 7.53 m from the centreline and 2.69 m below the surface. The collapse mechanism of the $\psi = 0$ soil is not known.

Table 1. The benchmark (Potts & Zdravković, 2001, §6.5.4.4; elastic constants p. 226).

Quantity	Value	Note
Friction angle φ'	25°	Mohr–Coulomb
Dilatancy angle ψ	0	plastic flow is isochoric
Cohesion c', unit weight γ	0, 0	cohesionless, weightless
Surcharge q_0	100 kPa	beside the footing
Footing	rigid strip, smooth	plane strain; width $B = 2$ m in our computations
Elastic constants (for elastic–plastic FE only)	$E' = 100$ MPa, $\nu = 0.3$	do not affect any bound in this report

1.3 What this report contributes

1. The two classical theorems stated with their hypotheses (§2), and the certified bracket $1 \le N_q \le 10.74$ they give, with the reasons domain truncation cannot invalidate the upper value (§3).
2. Proposition 1: for $c = 0$ and $\psi = 0$ no comparison material improves the lower value 1 — the theorem is exactly as weak here as it looks (§4.1).
3. Proposition 2: a lower bound conditional on a floor for the mean effective stress, with its values (§4.2).
4. Proposition 3: Davis's reduced-strength value is a lower bound within the class of statically determinate translational mechanisms, and only there (§4.3).
5. Every published and in-house number, placed on one axis with its status (§5), and the problems we could not close (§6).

2 Limit theorems for non-associated materials

2.1 Setting and notation

The soil is rigid–perfectly plastic and the problem is plane strain; stresses σ are effective and positive in tension, and only the in-plane components do work. With in-plane principal stresses $\sigma_1 \ge \sigma_3$, write $p = -(\sigma_1 + \sigma_3)/2$ for the mean effective stress (positive in compression) and $R = (\sigma_1 - \sigma_3)/2$ for the radius of the Mohr circle. The Mohr–Coulomb yield condition is

$$f(\sigma) = R - p\sin\varphi - c\cos\varphi \le 0\,. \tag{1}$$

Let $\mathbf{n}$ and $\mathbf{m}$ be the unit vectors of the major (most tensile) and minor in-plane principal directions. With $\psi = 0$ the plastic strain rate is coaxial with the stress and isochoric:

$$\dot\varepsilon = \lambda(\mathbf{n}\otimes\mathbf{n} - \mathbf{m}\otimes\mathbf{m}), \quad \lambda \ge 0, \quad \operatorname{tr}\dot\varepsilon = 0, \tag{2}$$

so that the rate of plastic work per unit area is

$$\sigma{:}\,\dot\varepsilon = R\dot\gamma, \quad \dot\gamma = \dot\varepsilon_1 - \dot\varepsilon_3 = 2\lambda \ge 0\,. \tag{3}$$

Velocity fields may carry tangential jumps $[v_t]$ across lines Γ (no normal jump, since $\psi = 0$). Treating a jump as the limit of a thin layer in simple shear, the principal axes in the layer lie at 45° to it and the shear traction on it equals R; the work rate per unit length is $R\,|[v_t]|$. For the benchmark the loading has two parameters: the footing pressure q and the surcharge q_0. A *collapse solution* is a pair (σ, v): a stress field in equilibrium with the loads and nowhere violating (1), and a velocity field obeying the flow rule wherever it deforms, with

the footing moving down at speed $V > 0$ and the velocity vanishing far away. Following Radenković ("postulons l'existence d'une solution"), we assume throughout that one exists; we call this hypothesis **H0**. Nothing in this report assumes that it is unique.

Lemma 1 (power identity). Virtual power; the identity behind Salençon (1977), App. V.A, Eqs. (26)–(28).
For any collapse solution of the benchmark with $\psi = 0$,

$$(q - q_0)\,BV = \int_\Omega R\dot{\gamma}\,\mathrm{d}A + \int_\Gamma R\,|[\![v_t]\!]|\,\mathrm{d}s \;\geq\; 0\,. \tag{4}$$

Proof. By virtual power, the power of the external loads equals the internal power. The footing contributes qBV. The surcharge acts as a pressure on the free surface beside the footing, which moves upward with some normal speed v_n, contributing $-q_0 \int v_\mathrm{n}\,\mathrm{d}x$. The flow is isochoric (2) and the velocity vanishes far away, so the net flux through the boundary is zero: the soil displaced under the footing, BV, leaves through the free surface, $\int v_\mathrm{n}\,\mathrm{d}x = BV$. The weightless soil carries no body force, and by (3) and the jump rule above the internal power is the right-hand side of (4), which is non-negative because $R \geq 0$ and $\dot{\gamma} \geq 0$. ∎

2.2 The two theorems

We state Radenković's theorems in the form Salençon gives them (1977, App. V.A §3, pp. 141–145), where they are proved for a general convex criterion. Radenković's note is written for a cone of Drucker–Prager type, $f = {J_2}^{1/2} + \alpha J_1 - K$; the Mohr–Coulomb case is Salençon's Remark 3.3.2(2). Let (M) be the real, non-standard system; (F) the same system made of a standard (associated) material with the same yield function f; and, for a loading $\mathbf{Q}$, let K_F denote the set of loads that some stress field in equilibrium and nowhere violating f can carry.

Theorem 1 (upper bound). Radenković (1961), result 1°; Salençon (1977), App. V.A §3.2, pp. 141–142, and Ch. V §8.1, p. 120.
Whatever its flow rule and elastic properties, and along every loading path, every load that (M) can carry — in particular every limit load — lies in K_F:

$$\mathbf{Q}_\mathrm{lim} \in K_M = K_F\,. \tag{5}$$

Hence any upper bound for the associated system (F), from the classical kinematic theorem, is an upper bound for every collapse load of (M).

Theorem 2 (lower bound). Radenković (1961), result 2°; Salençon (1977), App. V.A §3.3, Eqs. (24)–(29), pp. 143–144.
Suppose there is a function g, convex on stress space, such that

$$g(0) \leq 0; \quad \forall\sigma \text{ with } f(\sigma) = 0,\ \exists\sigma' \colon\ g(\sigma') = 0,\quad v^P(\sigma) \in \lambda\,\partial g(\sigma'),\ \lambda \geq 0, \tag{6}$$

$$(\sigma - \sigma') \colon v^P(\sigma) \geq 0, \tag{7}$$

where $v^\mathrm{P}(\sigma)$ is any plastic strain rate that the flow rule of (M) associates with σ. Let (G) be the system made of the standard material with yield function g and $P_\mathrm{G}(v)$ its dissipation. Then every collapse solution (σ, v) of (M) satisfies

$$P_G(v) - \mathbf{Q}(\sigma)\cdot\dot{\mathbf{q}}(v) \leq 0, \qquad \text{hence} \quad \mathbf{Q}_\mathrm{lim} \in K_F \setminus \mathring{K}_G, \tag{8}$$

where $\mathbf{Q}\cdot\dot{\mathbf{q}}$ is the power of the loads and $\mathring{K}_{\mathrm{G}}$ the interior of K_{G}: limit loads of (M) lie in the "ring" between the load sets of (G) and (F).

Remarks. (i) Theorem 2 holds for every admissible g, which is not unique (Salençon, Remark 3.3.2(3)); the best bound is the envelope of the sets K_{G} over all admissible g, as indicated by Palmer (Salençon §3.3.5, p. 145). (ii) For a Mohr–Coulomb material with a constant ψ, an admissible g is the Coulomb function with angle ψ and cohesion $c\tan\psi/\tan\varphi$ (Remark 3.3.2(2)). With $c = 0$ and $0 < \psi \le \varphi$, (G) is a cohesionless associated soil with friction angle ψ, whose weightless strip limit load is Prandtl's, so Theorem 2 gives $N_{\mathrm{q}} \ge N_{\mathrm{q}}{}^{\mathrm{P}}(\psi)$ in the notation of Eq. (9). (iii) For $\psi = 0$ and $c = 0$, this g describes a liquid, a material that carries isotropic stress only; Salençon notes that the inner bound is then "trivial" (§3.4, p. 145) and that the results are "too conservative to be of practical use, particularly in the case of $\psi = 0$" (Ch. V §8.3, p. 121). Proposition 1 below shows that no other choice of g does better.

3 The certified bracket

3.1 Upper bound: $N_{\mathrm{q}} \le 10.74$

By Theorem 1, no collapse load of the $\psi = 0$ soil exceeds the limit load of the same soil with associated flow, and any kinematically admissible velocity field for the associated soil bounds that from above. For the associated weightless soil the exact value is Prandtl's,

$$N_q^P(\varphi) = \frac{1+\sin\varphi}{1-\sin\varphi}\, e^{\pi\tan\varphi}, \qquad N_q^P(25°) = 10.662142\ \ldots \tag{9}$$

(Prandtl, 1920; Potts & Zdravković, 2001, p. 226, who note it is exact for $\psi = \varphi$ and both smooth and rough footings). We computed an upper bound independently, by finite-element limit analysis of the associated problem (the upper-bound formulation of Sloan, 1989, posed as a conic program as in Krabbenhøft, Lyamin & Sloan, 2007). The discrete velocity field satisfies the associated flow rule everywhere in every element and on every discontinuity, so the computed load is a rigorous upper bound once the program is solved to optimality. The result is

$N_{\mathrm{q}}{}^{\mathrm{UB}} = 10.734993$ — optimal, relative duality gap 8.4×10^{-8} — which is 0.683 % above Prandtl's value.

We report it as **10.74**, rounded upwards. Two facts show that truncating the half-space to a finite domain cannot have raised this number above a true bound. *First*, the computed velocity vanishes on the truncation boundary; extended by zero it is kinematically admissible for the half-space, so 10.735 bounds the half-space problem directly. *Second*, the truncation does not even change the exact associated answer: Prandtl's mechanism (Figure 1) reaches 7.53 m from the centreline and 2.69 m deep, inside the computational half-domain (10.80 m wide, 4.90 m deep), so the exact associated load of the truncated domain is also 10.6621. The 0.68 % excess is therefore the discretisation of the velocity field, not the boundary. The computation is in floating point, not exact arithmetic, as is standard for finite-element limit analysis.

Remark. Theorem 1 combined with the exactness of Prandtl's solution for the associated material gives the sharper $N_{\mathrm{q}} \le 10.662$. That route rests on the classical construction of an admissible extension of Prandtl's stress field to the whole half-space (Salençon, 1977, Ch. V §8.2, p. 120, gives the references). We quote 10.74 as the certified value because it is the bound we computed and checked ourselves; it does not depend on that construction.

3.2 Lower bound: $N_q \geq 1$

Corollary 1. CERTIFIED · EXACT
Under H0, every collapse load of the benchmark with $\psi = 0$ satisfies $q \geq q_0$, i.e. $N_q \geq 1$.

Proof. Immediate from Lemma 1, since $V > 0$. Equivalently, Theorem 2 with the liquid g of Remark (iii): $P_G(v) = 0$ for every isochoric field with tangential jumps, and (8) reduces to $(q - q_0)BV \geq 0$ (Salençon, 1977, Eq. (28) and §3.4). The argument uses no numerics, holds equally with gravity, and holds on a truncated domain. ∎

3.3 The statement

For the benchmark with $\psi = 0$ the certified statement is $\mathbf{1 \leq N_q \leq 10.74}$. The interval is wide because the physics is: Section 4.1 shows that the classical lower-bound theorem cannot narrow it, and no other theorem we know of applies without an assumption about how the soil collapses.

4 New results

4.1 The lower value 1 cannot be improved by the theorem

Proposition 1. PROVED
Let $c = 0$ and $\psi = 0$. If a convex g satisfies the hypotheses (6)–(7) of Theorem 2 at every point of the yield surface $f = 0$, then every stress in $G = \{g \leq 0\}$ has zero in-plane deviator: (G) is a liquid. Consequently $P_G(v) = 0$ for every admissible $\psi = 0$ velocity field, inequality (8) is equivalent to $N_q \geq 1$ for every such g, and the Palmer envelope over all of them gives no more.

Proof. Take σ on the yield surface with principal directions $\mathbf{n}$, $\mathbf{m}$. By (2) its plastic strain rates are the positive multiples of $\mathbf{d} = \mathbf{n}\otimes\mathbf{n} - \mathbf{m}\otimes\mathbf{m}$. By (6), $\mathbf{d}$ is an outward normal of G at σ', so $h_G(\mathbf{d}) := \sup_{s\in G} s{:}\mathbf{d} = \sigma'{:}\mathbf{d}$, and by (7) $\sigma'{:}\mathbf{d} \leq \sigma{:}\mathbf{d}$. Writing $\sigma = -p\mathbf{I} + R\mathbf{d}$ with $R = p\sin\varphi$ on the yield surface ($c = 0$), and using $\mathbf{I}{:}\mathbf{d} = 0$ and $\mathbf{d}{:}\mathbf{d} = 2$, gives

$$h_G(\mathbf{d}) \leq \sigma{:}\,\mathbf{d} = 2p\sin\varphi\,. \tag{10}$$

The hypotheses must hold at every point of the yield surface, and the cone $f = 0$ reaches its apex: for each orientation the stress $-p\mathbf{I} + p\sin\varphi\,\mathbf{d}$ lies on it for every $p > 0$, with the same flow direction $\mathbf{d}$. Letting $p \to 0$ in (10) gives $h_G(\mathbf{d}) \leq 0$. Exchanging the roles of $\mathbf{n}$ and $\mathbf{m}$ (the stress rotated by 90°, also on the cone) gives $h_G(-\mathbf{d}) \leq 0$. Hence $s{:}\mathbf{d} = 0$ for every $s \in G$ and every orientation; and as $\mathbf{n}$ turns, $\mathbf{d} = [\cos 2\theta, \sin 2\theta; \sin 2\theta, -\cos 2\theta]$ spans the in-plane deviatoric tensors. So every $s \in G$ is isotropic in the plane (G is non-empty because $g(0) \leq 0$). For such s, $s{:}\dot{\varepsilon} = 0$ for every isochoric strain rate and $s\mathbf{n}\cdot[v] = 0$ for every tangential jump, so $P_G(v) = 0$ and (8) reads $(q - q_0)BV \geq 0$. In a weightless liquid the stress is a uniform pressure, so K_G contains only loads with $q = q_0$; the envelope of such sets adds nothing. ∎

Salençon called the theorem "too conservative … particularly in the case of $\psi = 0$"; Proposition 1 shows the diagnosis is exact, not a matter of choosing g badly. Any sharper lower bound must use information the theorem does not: an a priori range for the stresses actually reached, which Salençon's §3.4 already suggests, or an assumption about the mechanism. The next two propositions take one route each.

4.2 A bound conditional on a stress floor

Proposition 2. CONDITIONAL
Assume H0 and suppose that in the collapse solution $R \geq k > 0$ wherever the soil deforms (in Ω where $\dot{\gamma} > 0$ and on every line where $[v_t] \neq 0$). Then $q \geq q_0 + (2+\pi)k$. For $c = 0$, where $R = p \sin\varphi$ at yield, the hypothesis $p \geq p_{\min}$ wherever the soil deforms gives

$$N_q \geq 1 + (2+\pi)\sin\varphi \,\frac{p_{\min}}{q_0}\,. \tag{11}$$

Proof. Let $\Gamma(v) = \int_\Omega \dot{\gamma}\,\mathrm{d}A + \int_\Gamma |[v_t]|\,\mathrm{d}s$. The actual collapse velocity is isochoric, has only tangential jumps and meets the footing and far-field conditions, so it is kinematically admissible for the same footing on a Tresca material of cohesion k, whose dissipation is $k\dot{\gamma}$ per unit area and $k|[v_t]|$ per unit length. Prandtl's solution for that material is exact, $q_T = q_0 + (2+\pi)k$ (Prandtl, 1920: the $\varphi \to 0$ limit of his N_c, as given by Potts & Zdravković, 2001, p. 226), so the kinematic theorem gives $(q_T - q_0)BV \leq k\Gamma(v)$, that is

$$\Gamma(v) \geq (2+\pi)\,BV\,. \tag{12}$$

By Lemma 1 and the hypothesis, $(q - q_0)BV = \int R\,\mathrm{d}\Gamma \geq k\Gamma(v) \geq (2+\pi)kBV$. Dividing by $BV > 0$ proves the first claim; putting $k = p_{\min}\sin\varphi$ and dividing by q_0 gives (11). This is Theorem 2 with g a Tresca criterion of cohesion k, used in the restricted form of Salençon's §3.4 where (6)–(7) are needed only for the stresses actually reached. ∎

Remark. The proof uses the hypothesis only through $\int R\,\mathrm{d}\Gamma \geq k\Gamma(v)$. It is therefore enough that the deformation-weighted mean of R over the collapse field be at least k; a pointwise floor is sufficient, not necessary.

Table 2. Proposition 2 for φ = 25°: the bound for three reference stress levels, and the stress floor each reference value would need. (2 + π) sin 25° = 2.17293.

Stress floor $p_{\min}$	$p_{\min}/q_0$	Bound on N_q	Meaning of the level
$q_0/(1+\sin\varphi)$	0.703	2.527	limiting state whose major principal stress is q_0
q_0	1.000	3.173	the surcharge itself
$q_0/(1-\sin\varphi)$	1.732	4.763	limiting state whose minor principal stress is q_0 (Prandtl's passive zone)
STRESS FLOOR NEEDED FOR THE BOUND TO REACH			
Davis, 8.58	3.49	8.58	—
Potts & Zdravković, 10.03	4.16	10.03	—
Prandtl, 10.66	4.45	10.66	—

Status. The hypothesis concerns the unknown collapse field. Our own computed collapse fields show the mean stress falling close to zero at the footing edge, so a pointwise floor does not hold there; we claim none of these values for this footing. Table 2 is useful in the other direction: for a stress-floor argument to certify a value near the published estimates, the soil would have to stay at more than four times the surcharge everywhere it deforms.

4.3 Davis's value is a bound within the translational class only

Following Davis (1968), Drescher & Detournay (1993, pp. 448–449, Eqs. 12–16) show for translational mechanisms that on a velocity discontinuity of a material with a coaxial non-associated flow rule the tractions satisfy a Coulomb relation with reduced strength,

$$\tan\varphi^{*} = \eta\tan\varphi, \quad c^{*} = \eta c, \quad \eta = \frac{\cos\psi\cos\varphi}{1-\sin\psi\sin\varphi}. \tag{13}$$

For $\psi = 0$, $\eta = \cos\varphi$ and $\tan\varphi^{*} = \sin\varphi$: at $\varphi = 25°$, $\eta = 0.906308$, $\varphi^{*} = 22.9098°$ and $N_q^{\mathrm{P}}(\varphi^{*}) = 8.5814$.

Proposition 3. CONDITIONAL
Suppose (hypothesis T) that the $\psi = 0$ soil collapses by a mechanism of rigid blocks in translation separated by plane velocity discontinuities, that the mechanism is statically determinate (each block has as many unknown forces as equilibrium equations), and that the same block geometry admits a kinematically admissible translational mechanism for the associated material with friction angle φ^{*}. Then its collapse load satisfies $N_q \geq N_q^{\mathrm{P}}(\varphi^{*}) = 8.58$.

Proof. On every discontinuity of the mechanism the tractions satisfy (13) (Drescher & Detournay, Eq. 12, with $c^{*} = 0$). The mechanism being statically determinate, the block forces — and with them the footing load — follow from equilibrium and this traction relation alone; the associated material with friction angle φ^{*} has the same traction relation on the same lines, so the same calculation gives the same load. Drescher & Detournay show that force equilibrium and the energy balance are equivalent for translational mechanisms of any rigid–perfectly plastic material (pp. 448–449 and Appendix 1), so this load is the energy-balance load of the corresponding mechanism of the φ^{*} material, which is kinematically admissible by hypothesis. By the kinematic theorem for that standard material, the load is at least its exact limit load, which for the weightless strip is Prandtl's $N_q^{\mathrm{P}}(\varphi^{*})$ by (9). ∎

Status. Hypothesis T cannot be checked for the real collapse, whose mechanism is unknown. Drescher & Detournay interpret the reduced-strength load as the state in which every shear layer has reached its residual strength and add: "Since the true failure mechanism is more complex, and not all shear layers reach the residual strength or fewer develop, the actual limit load may be higher" (p. 451); they call it "certainly a safe engineering estimate of the true limit load" (p. 453). Proposition 3 makes the scope of the word "safe" exact: Davis's 8.58 is a lower bound within the translational class, and outside it a useful engineering estimate, not a bound.

5 Estimates and how they compare

Table 3 lists every number for this benchmark that we know of, with its source and status, and Figure 2 places them on one axis.

Table 3. Every value for the benchmark ($\varphi' = 25°$, $\psi = 0$, weightless, cohesionless, smooth unless stated).

Result	N_q	Status	Source
Upper bound — associated flow, finite-element limit analysis	≤ 10.74	CERTIFIED	§3.1 (10.734993)
Lower bound — Radenković's theorem	≥ 1	CERTIFIED · EXACT	§3.2; Salençon 1977 §3.4
Associated-flow exact value (Prandtl)	10.662	reference ($\psi = \varphi$ only)	Eq. (9)
Davis reduced strength, $\tan\varphi^{*} = \sin\varphi$	8.581	CONDITIONAL	Prop. 3; D&D 1993 Eqs. 12–16

Result	N_q	Status	Source
Stress floor $p_{\min} = q_0$	3.173	CONDITIONAL	Prop. 2; not met at the footing edge
Potts & Zdravković FE, $\psi = 0$, smooth	10.03	ESTIMATE	P&Z 2001 Table 6.1, p. 232
Potts & Zdravković FE, $\psi = 0$, rough	9.92	ESTIMATE	P&Z 2001 Table 6.1, p. 232
Loukidis & Salgado fit, Eqs. (8)–(9), at $\varphi = 25°$, $\psi = 0$	10.10	ESTIMATE	extrapolated: fit covers $\varphi = 30°$–$45°$
TerraBrains regularised FE, two independent settings	10.3 – 10.6	ESTIMATE	this report, §5
For calibration: P&Z FE with $\psi = \varphi = 25°$, smooth / rough	10.8 / 11.03	estimate of 10.66	P&Z 2001 Table 6.1; errors 0.9 % / 3 %, p. 227

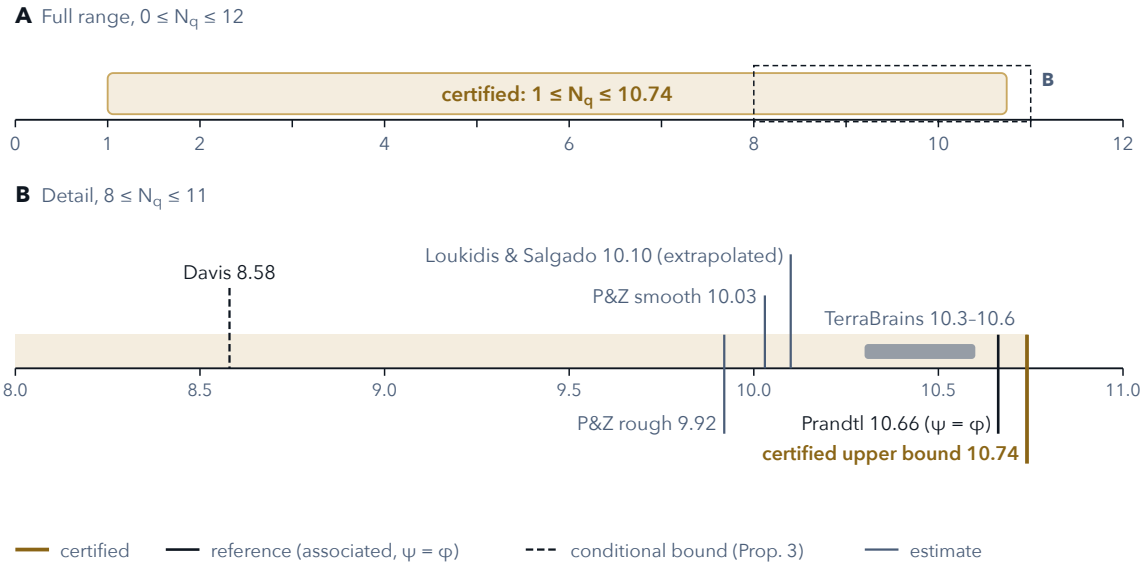


Figure 2. The result on one N_q axis. Shaded: what is proved, $1 \le N_q \le 10.74$. Panel B enlarges $8 \le N_q \le 11$, where every other number lies. Solid gold: certified; solid dark: the exact associated value, which is not a bound for $\psi = 0$ from below; dashed: Davis's value, a bound only under hypothesis T; grey: estimates. Potts & Zdravković values from their Table 6.1 (p. 232); the Loukidis & Salgado value is their fit extrapolated below its range.

Published finite-element values. Potts & Zdravković analysed the benchmark with an elastic–plastic Mohr–Coulomb model under displacement control. Their three $\psi = 0$ analyses "give a value of N_q of approximately 10.0. This is some 7 % lower than" Prandtl's value (p. 227); Table 6.1 gives 10.03 (smooth) and 9.92 (rough). Their associated analyses exceed the exact 10.66 by 0.9 % and 3 %, and note that a finer mesh near the footing edge improves the predictions. By Theorem 1, which holds whatever the elastic properties and loading path (Salençon, 1977, p. 120), the exact $\psi = 0$ load of their model cannot exceed 10.66; their computed values do not.

A fitted formula. Loukidis & Salgado (2009, p. 877) fitted their finite-element results by $N_q = (1 + \sin\varphi)/(1 - \sin\varphi) \cdot \exp(F\,\pi \tan\varphi)$ with $F = 1 - \tan\varphi\,[\tan(0.8(\varphi - \psi))]^{2.5}$. At $\varphi = 25°$, $\psi = 0$ it gives 10.0956. Their analyses cover $\varphi = 30°$–$45°$, so this is an extrapolation and is listed as such.

TerraBrains estimates. TerraBrains regularised finite-element estimates at two independent settings give 10.3 – 10.6; weaker settings give lower values. Run on the associated case, where the exact answer is known, the same two settings come out 2 – 8 % high, so 10.3 – 10.6 should be read as an upper-leaning estimate. We do not claim a limit as the settings are weakened; on affordable meshes our estimates do not converge to one.

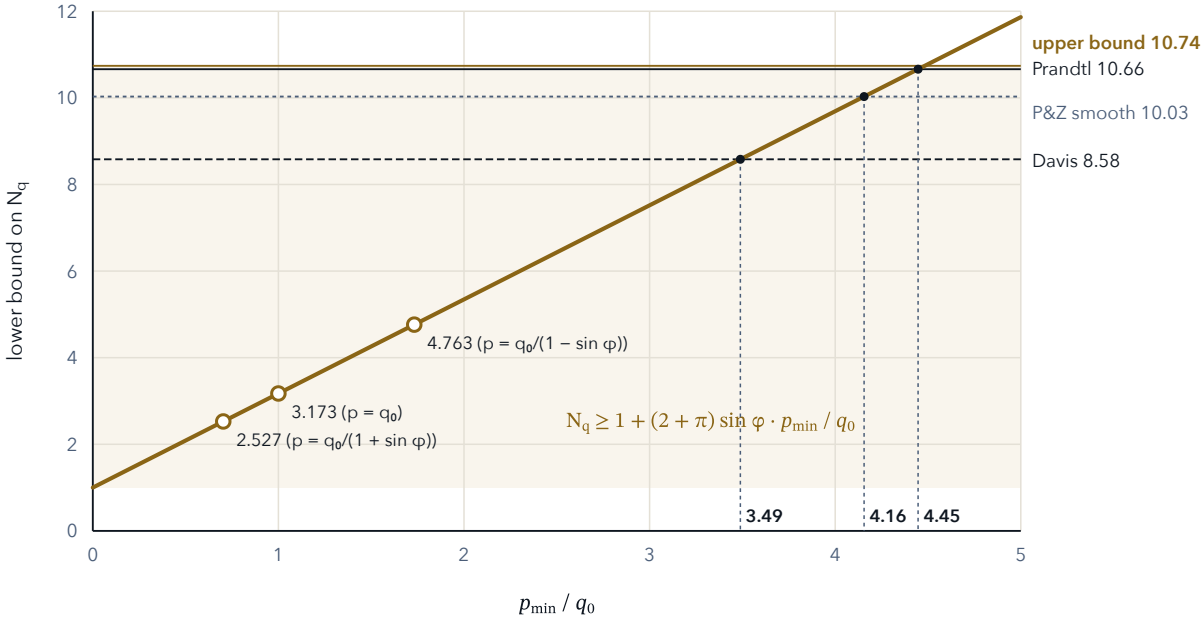


Figure 3. Proposition 2: the lower bound on N_q as a function of an assumed floor p_{min} on the mean effective stress in the deforming soil ($\varphi = 25°$). Open circles: the three reference stress levels of Table 2. Filled dots and the values on the axis: the floor that would be needed to reach Davis's 8.58, Potts & Zdravković's 10.03 and Prandtl's 10.66. The light band is the certified bracket.

Reading the comparison. All the estimates sit in the top sixth of the certified interval, between 9.9 and 10.6, and they agree with each other to within about ±3.5 %. That agreement is evidence about the methods, not a proof about the soil: Proposition 1 shows the classical theorem cannot support any of them, Proposition 2 would need a stress floor above four times the surcharge, and Proposition 3 would need a mechanism of sliding blocks — and even then would give only 8.58. An engineer who needs a number with a proof behind it has $1 \leq N_q \leq 10.74$, or Davis's 8.58 under an explicit assumption.

6 Open problems

1. **An assumption-free lower bound above 1.** What is the infimum of the possible $\psi = 0$ collapse loads? Proposition 1 rules out Radenković's theorem as stated; a proof must use information about the stresses actually reached or about the mechanism. One route we are pursuing integrates equilibrium along curves that cross the deforming zone; it needs control of how slip is distributed, which we do not yet have. As far as we can find, no such bound is published.
2. **The limit of regularised estimates.** Whether that limit exists, whether it is unique, and whether it is a collapse load of the original problem are open. We know of no published convergence proof for $\psi < \varphi$, and none of our numbers depends on one.
3. **Error bars for a single estimate.** The non-associated problem is not variational, so the a-posteriori error estimates available for associated limit analysis do not apply. No published method gives a rigorous error bar for one finite-element value of the $\psi = 0$ collapse load.

4. **Rough footings.** Theorem 1 applies unchanged, and Prandtl's value is exact for the associated rough footing (Potts & Zdravković, 2001, p. 226). A computed rough-footing upper bound as tight as the smooth one, and any rough-footing lower bound above 1, remain to be done.

7 Conclusions

- For the Potts & Zdravković benchmark ($\varphi' = 25°$, $\psi = 0$, weightless, cohesionless, smooth strip) the certified statement is $\mathbf{1 \le N_q \le 10.74}$: an upper bound from finite-element limit analysis of the associated problem, 0.68 % above Prandtl's exact 10.662, and the exact lower bound of Radenković's theorem.
- The lower value cannot be raised by Radenković's theorem with any comparison material: for $c = 0$ and $\psi = 0$ every admissible one is a liquid (Proposition 1). Salençon's remark that the theorem is too conservative for $\psi = 0$ is exact.
- Two lower bounds hold under stated hypotheses: $1 + (2 + \pi) \sin \varphi \cdot p_{\min}/q_0$ under a stress floor (Proposition 2; 3.173 at $p_{\min} = q_0$), and Davis's 8.58 for statically determinate translational mechanisms (Proposition 3). Neither hypothesis is known to hold for this footing, and neither value is claimed for it.
- Every computed or fitted value — Potts & Zdravković's 10.03 and 9.92, Loukidis & Salgado's extrapolated 10.10, TerraBrains' 10.3 – 10.6 — lies inside the bracket and is an estimate.
- Whether any bound above $N_q = 1$ holds with no assumption about the collapse mechanism is, as far as we can find, open.

On TerraBrains. The certified bearing-capacity analysis returns, for a rigid strip on Mohr–Coulomb soil with $0 \le \psi \le \varphi$ (any cohesion, unit weight and surcharge), the certified bracket and the problem domain it holds for, the Davis estimate, the TerraBrains estimates with whether each is mesh-converged, and the exact associated value where one exists. It runs on the TerraBrains compute tier from any project's Bearing capacity analysis (Certified bracket).

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A Closed-form values

Computed in 30-digit arithmetic (mpmath) and rounded; each can be checked on a calculator from the formula shown.

Table A1. Closed forms used in this report, φ = 25°.

Quantity	Formula	Value
Prandtl, associated	$N_q^{\mathrm{P}}(\varphi)$, Eq. (9)	10.662142388
Davis factor, $\psi = 0$	$\eta = \cos\varphi$	0.906307787
Davis reduced angle	$\varphi^* = \arctan(\sin\varphi)$	22.909807°
Davis value	$N_q^{\mathrm{P}}(\varphi^*)$	8.581430
Loukidis & Salgado fit, $\psi = 0$	Eqs. (8)–(9), see §5	10.095639
Stress-floor slope	$(2 + \pi)\sin\varphi$	2.172931
Proposition 2, $p_{\min} = q_0/(1 + \sin\varphi)$	Eq. (11)	2.527417
Proposition 2, $p_{\min} = q_0$	Eq. (11)	3.172931
Proposition 2, $p_{\min} = q_0/(1 - \sin\varphi)$	Eq. (11)	4.763422
Upper-bound excess over Prandtl	10.734993 / 10.662142 − 1	0.683 %
Prandtl mechanism extent, $B = 2$ m	active wedge + fan + passive wedge	7.53 m × 2.69 m

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